CONSTRAINTS ON THE GEOMETRIES OF BLACK HOLES IN CLASSICAL AND SEMICLASSICAL GRAVITY

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Constraints on the geometries of static spherically symmetric black holes are obtained by requiring that the spacetime curvature be analytic at the event horizon. Further constraints are obtained by requiring that the semiclassical trace equation be satisfied in the case that only conformally invariant fields are present. It is found that there exists a range of sizes for which zero temperature black holes do not exist. The range depends on the number and types of quantized fields present.

It is well known that the stress-energy of quantized fields can significantly alter the spacetime geometry near the event horizon of a black hole. However, due to the difficulties involved in computing the stress-energy of quantized fields in black hole spacetimes, only the linearized semiclassical backreaction equations have been solved. Thus it is useful to try and see if constraints can be placed on black hole solutions to the full nonlinear semiclassical backreaction equations.

Previously Mayo and Bekenstein [1] looked at the problem of constraining static black hole solutions to Einstein's equations for various types of matter fields. They found that black hole solutions exist if the stress-energy tensor is finite on the horizon and one component satisfies a certain inequality. The geometry near the horizon is of the same form as the Schwarzschild geometry except in the limit of an extreme black hole where the inequality becomes an equality. In this latter case they put constraints on the form of the geometry near the horizon. They also point out that if the transition between nonextreme metrics and an extreme metric is thought of as analogous to a phase transition, then it is likely that the metric of an extreme black hole is of the same form near the horizon as the extreme Reissner-Nordström black hole. Otherwise the transition would correspond to a third or higher order phase transition and such transitions have not been observed in nature. Recently Zaslavskii [2] has used a power series expansion of the metric to determine the general form it takes near the horizons of near extreme and extreme charged black holes in a cavity when the grand canonical ensemble is utilized.

In this paper we take a different approach and first find constraints on the geometries of static spherically symmetric black holes by imposing reasonable constraints on the spacetime curvature at their event horizons. The results are valid for any classical or semiclassical metric theory of gravity. We find that nonzero temperature black holes must have geometries of the same form as the Schwarzschild geometry near their event horizons. Then we specialize to semiclassical gravity when only conformally invariant fields are present. By requiring that the trace of the semiclassical backreaction equations be satisfied we find that there exists a range of sizes for which zero temperature black holes cannot exist. The size of the range depends upon the number and types of quantized fields present.

Our constraint on the curvature is to require that the curvature be analytic at the event horizon in terms of the radial coordinate r, where the proper area of a two-sphere centered on the origin is $4\pi r^2$. The assumption of analyticity, while strong, is not without precedent. Hawking [3] assumed an analytic metric at the event horizon in one of his uniqueness theorems for stationary rotating black holes.

To begin we write the metric for a static spherically symmetric spacetime in the general form

$$ds^2 = -f(r)dt^2 + \frac{1}{k(r)}dr^2 + r^2 d\Omega^2 \quad . \tag{1}$$

The unique non vanishing components of the Riemann curvature tensor in an orthonormal frame are

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = \frac{v'k}{2} + \frac{vk'}{4} + \frac{v^2 k}{4} \tag{2}$$

$$R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = \frac{vk}{2r} \tag{3}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -\frac{k'}{2r} \tag{4}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1-k}{r^2} \quad , \tag{5}$$

where $v \equiv f'/f$ and primes denote derivatives with respect to r. The Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is proportional to the sum of the squares of these components. If the spacetime has an event horizon then f vanishes on that horizon. The surface gravity at the event horizon is

$$\kappa = \frac{v}{2}\,(fk)^{1/2} \quad . \tag{6}$$

We require that the above components of the Riemann tensor be analytic at the event horizon. It is clear from Eq.(5) that k must be analytic at the horizon. From Eq.(3) it is seen that the quantity $v\,k$ must also be analytic. This second condition results in the further condition that $k \to 0$ at the event horizon. To see this note that v must diverge at the horizon because $v = f'/f$ and f vanishes there. These conditions can be summarized by saying that near the event horizon v and k have the following leading order behaviors:

$$\begin{aligned} v &= p(r-r_0)^{m-n} \\ k &= q(r-r_0)^{n} \quad . \end{aligned} \tag{7}$$

Here p and q are real positive constants, m and n are integers which satisfy the condition $n > m \geq 0$, and r_0 is the value of r at the event horizon.

Further restrictions can be obtained from Eq.(2). Substituting Eqs.(7) into (2) one finds that the terms on the right hand side of (2) must either be separately finite at the horizon or they must cancel. They only cancel if $m = n-1$ and $p = 2-n$. However we previously showed that $n \geq 1$. Thus, since $p > 0$, the only case in which they cancel is $n = 1,\ m = 0,\ \ p = 1$. In this case it is easy to see that near the horizon $f = c\,(r-r_0)$ for some $c > 0$. The surface gravity is nonzero in this

case so these solutions describe nonzero temperature black holes. If the terms in (2) are separately finite then the restrictions are $m \geq 1$ and $2m \geq n$. Examination of Eq.(6) shows that such solutions describe zero temperature black holes.

Semiclassical gravity can be used to place further constraints on the geometry in the case that only conformally invariant free quantized fields are present. In this case the trace of the semiclassical backreaction equations is

$$- R - 6a\Box R = 8\pi[\alpha\Box R + \beta(R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3}R^2) + \gamma C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}] \tag{8}$$

where a is the coefficient of an R^2 term in the gravitational Lagrangian and α, β, and γ are determined by the numbers and types of quantized fields present [4].

If Eqs.(7) are substituted into Eq.(8) and the above conditions on m and n are imposed, then the leading order terms near the horizon can be computed for various values of m and n. Consider first the case $2m > n > m \geq 2$. In the limit $r \to r_0$ Eq.(8) becomes

$$-\frac{16\pi}{3{r_0}^2}(\beta + 2\gamma) - \frac{2}{{r_0}^2} = 0 \tag{9}$$

For all fields $\beta + 2\gamma > 0$ [4]. Thus there are no solutions to the trace equation for values of m and n in this range.

The only other possibility is $2m = n \geq 2$. If Eqs.(7) are substituted into Eq.(8) for values of m and n in this range, the limit $r \to r_0$ is taken, and the resulting equation is solved for q, the result is

$$q_{\pm} = \frac{3{r_0}^2 - 32\pi(\beta - \gamma) \pm (768\pi^2\beta^2 - 3072\pi^2\beta\gamma - 288\pi\beta{r_0}^2 + 9{r_0}^4)^{1/2}}{4\pi(\beta + 2\gamma)p^2{r_0}^2} \; . \tag{10}$$

Since the left hand side of (10) is positive and real, the right hand side must be also. However the right hand side is complex if $r_- < r_0 < r_+$ where

$$r_{\pm} = 4(\pi\beta)^{1/2}\left[1 \pm \left(\frac{2}{3\beta}\right)^{1/2}(\beta + 2\gamma)^{1/2}\right]^{1/2} \; . \tag{11}$$

For all allowed values of β and γ r_+ is real. If $\beta < 4\gamma$ then r_- is imaginary and solutions only occur for $r_0 > r_+$. If $\beta > 4\gamma$ then solutions also occur for $0 < r_0 < r_-$. Thus in all cases there is a range of values of r for which zero temperature black hole solutions to the semiclassical backreaction equations do not exist.

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References

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